Port-Hamiltonian formulation of shallow water equations with coriolis force and topography*

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1 Introduction

Port based network modeling of complex lumped parameter physical systems naturally leads to a generalized Hamiltonian formulation of its dynamics. The resulting class of open dynamical systems are called "Port-Hamiltonian systems" [12] which are defined using a Dirac structure, the Hamiltonian and dissipative elements. This formulation has been successfully extended to classes of distributed parameter systems by introducing an infinite-dimensional Dirac structure based on Stokes' Theorem [11]. The port-Hamiltonian formulation is a non-trivial extension of the Hamiltonian formulation. The key extension is the non-zero energy flow through the boundary of the spatial domain by using Dirac structures as opposed to the Poisson structures in the standard Hamiltonian formulation which assume zero energy flow through the boundary.

Port-Hamiltonian formulation can be applied to infinite-dimensional fluid dynamical systems through which governing equations emerge from basic conservation laws and follow additional conservation laws. For example, port-Hamiltonian formulation of shallow water flows gives rise to partial differential equations arising from mass and momentum conservation laws and naturally (*because of the Dirac structure*) derive additional conservation laws of energy, potential vorticity and enstrophy. Many numerical methods such as finite difference, finite volume and (dis)continuous Galerkin finite element methods have been developed to approximate these infinite-dimensional systems into finite-dimensional systems. Such finite-dimensional systems may satisfy basic conservation laws of the original system to the discrete level but they do not satisfy accurately the additional conservation laws. Such numerical methods for shallow water equations are already developed (see [2, 3, 10, 1]. In this paper, we therefore aim to develop a novel numerical approach by approximating the shallow water equations based on port-Hamiltonian formulation such that all conservation laws are satisfied to the discrete level.

Finite-dimensional port-Hamiltonian approximation of one-dimensional shallow water equations is presented. We first consider the infinite-dimensional port-Hamiltonian formulation of shallow water equations with respect to a non-constant Dirac structure. The discretization essentially consists of two parts: First we discretize the interconnection structure and next we discretize the constitutive relations of the energy storage. In the current work we consider linear shallow water equations, i.e. linearize the system around a given height and zero velocities. Given the port-Hamiltonian formulation of distributed parameter systems it is natural to use *different* finite-elements for the ap-

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proximation of functions and forms. In [5] this method was used for discretization of the ideal transmission line and the two dimensional wave equation. We also further extend the procedure in [5], by explaining how to handle various boundary conditions and in particular periodic boundary conditions. In addition we also overcome the constraint of requiring extremely small time steps by making use of an implicit scheme, which has no time step restriction. In this paper we apply this method to a special case of shallow water equations, which is a 1D port-Hamiltonian system defined with respect to a non-constant Stokes-Dirac structure.

2 Shallow water equations

We consider here, a special case of, the flow of water through an open channel canal. The flow dynamics can then be described by the shallow water equations [8] given as

$$\begin{aligned} \partial_t \tilde{h} + \partial_x(\tilde{h}\tilde{u}) &= 0 \\ \partial_t \tilde{u} + \partial_x(\frac{1}{2}\tilde{u}^2 + g(\tilde{h} + h_b(x))) - f\tilde{v} &= 0, \\ \partial_t \tilde{v} + \tilde{u}\partial_x \tilde{v} + f\tilde{u} &= 0, \end{aligned} \tag{1}$$

with initial conditions $\tilde{h}(x,0), \tilde{u}(x,0), \tilde{v}(x,0)$ and boundary conditions which can be given or time varying or the boundary can be periodic. These equations are essentially of the two-dimensional shallow water equations, where the variables are considered to be invariant in one of the flow directions. In the two dimensional case $\tilde{h}(x,y,t)$ denotes the height of the water level, $\tilde{u}(x,y,t)$ and $\tilde{v}(x,y,t)$ are the components of the water water velocities in the x and y directions respectively. The term $h_b(x,y)$ takes into account the topography and f is the Coriolis parameter. For our current analysis we assume that the water height, the velocity components and the topographic terms are invariant in the y direction. As a result of this assumption we only see the x dependency in equation (1) and hence we have a special case of shallow water equations. (Such models have also been derived and analyzed in [6]). Further, g is the acceleration due to gravity, x the spatial variable representing the length of the canal, *i.e.*, $x \in [0, L]$ and t the time. The first equation expresses the mass-balance and the second and third equation come from the momentum-balance. The total energy (Hamiltonian) of the system is

$$\mathcal{H} = \int_0^L [\frac{1}{2}(\tilde{h}\tilde{u}^2 + \tilde{h}\tilde{v}^2 + g\tilde{h}^2) + g\tilde{h}h_b]dx. \tag{2}$$

2.1 Notation

In this section we introduce a few notations and some mathematical background, which would help us build the port-Hamiltonian model of the shallow water equations. We apply the differential geometric framework of differential forms on the spatial domain $X \in [0, L]$ of the system. The shallow water equations considered in this paper is a case of a distributed parameter system with a one-dimensional spatial domain and in this context it means that we distinguish between zero-forms (functions) and one forms defined on the interval representing the spatial domain of the canal. One forms are objects which can be integrated over every sub-interval of the interval (*line integrals*) where as zero-forms *or functions* can be evaluated at any points of the interval. If we consider a spatial coordinate x for the interval X, then a zero-form is simply given by the values $f(x) \in \mathbb{R}$ for every coordinate value in x in the interval, while a one-form g is given as $\tilde{g}(x)dx$ for a certain density function g. We denote the set of zero forms and one-forms on X by $\Omega^0(X)$ and $\Omega^1(X)$ respectively. Given a coordinate x for the spatial domain we obtain by spatial differentiation of a function $f(x)$ the one-form $\omega := \frac{df}{dx}(x)dx$. In coordinate free language this is denoted as $\omega = \mathrm{d}f$, where d is called the exterior derivative mapping zero forms to one forms. We denote by $*$, the Hodge star operator mapping k forms to $n-k$ forms, meaning that given a one-form g on X, the star operator converts the one form g to a function $\tilde{g}$, mathematically given as $*g(x) = \tilde{g}(x)$. In the case of the current

paper $n = 1$ and k is either 0 or 1. Also denote by $\wedge$, the wedge product of two differential forms. Given a k-form ω_1 and an l-form ω_2, the wedge product $\omega_1 \wedge \omega_2$ is a $k + l$-form.

2.2 The port-Hamiltonian model

In the port-Hamiltonian formulation the basic concept we need is that of a Dirac structure, which is a generalization of symplectic and Poisson structures and is a geometric notion of formalizing a power conserving interconnections, see [12] and references therein for detailed explanations.

Definition 1. *Let V be a linear space (possibly infinite-dimensional). There exists on $V \times V^*$ the canonically defined symmetric bilinear form*

$$\ll (f_1, e_1), (f_2, e_2) \gg := < e_1 \mid f_2 > + < e_2 \mid f_1 > \tag{3}$$

with $f_i \in V, e_i \in V^*, i = 1, 2$ and $<|>$ denoting the duality product between V and its dual subspace V^*. A constant Dirac structure on V is a linear subspace $D \subset V \times V^*$ such that

$$D = D^{\perp}, \tag{4}$$

where $\perp$ denotes the orthogonal complement with respect to the bilinear form $\ll, \gg$.

Let now $(f, e) \in D = D^{\perp}$. Then as an immediate consequence of (3)

$$0 = \ll (f, e), (f, e) \gg = 2 < e \mid f > .$$

Thus for all $(f, e) \in D$ we have $< e \mid f >= 0$, expressing power conservation with respect to the dual power variables $f \in V$ and $e \in V^*$

The Stokes-Dirac structure corresponding to the 1D shallow-water equations (1) is defined as follows: The spatial domain $X \subset \mathbb{R}$ is represented by a one dimensional manifold with point boundaries. The energy exchange of the system with the environment takes place through the boundary $\{0, L\}$ of the system.The height of the water flow (representing the mass density) through the canal $h(x,t)(= \tilde{h}(x,t)dx)$ is identified with a 1-form on X. Note that the integral of h over a subinterval denotes the total amount of water contained in that subinterval. Furthermore, assuming the existence of a *Riemannian metric* $<,>$ on Z, we identify (by index raising w.r.t this Riemannian metric) the Eulerian vector fields u, v on Z with a 1-form. The energy variables are then the height $h(x,t)(= \tilde{h}(x,t)dx)$ and the velocity components $u(x,t)(= \tilde{u}(x,t)dx)$ and $v(x,t)(= \tilde{v}(x,t)dx)$. This leads to the consideration of the (linear) space of energy variables

$$\Omega^1(X) \times \Omega^1(X) \times \Omega^1(X).$$

To identify the boundary variables we consider space of 0-forms, i.e., space of functions on ∂X, to represent both the boundary mass flow and the dynamic pressure at the boundary. We thus consider the space of boundary variables

$$\Omega^0(\partial X) \times \Omega^0(\partial X) \times \Omega^0(\partial X).$$

Proposition 2. *Let $X \subset \mathbb{R}$ be a 1 dimensional manifold with boundary ∂X. Consider $V = \Omega^1(X) \times \Omega^1(X) \times \Omega^1(X) \times \Omega^0(\partial X)$ and $V^* = \Omega^0(X) \times \Omega^0(X) \times \Omega^0(X) \times \Omega^0(\partial X)$, together with the bilinear form*

$$\begin{aligned} &<< (f_h^1, f_u^1, f_v^1, f_b^1, e_h^1, e_u^1, e_v^1, e_b^1), (f_h^2, f_u^2, f_v^2, f_b^2, e_h^2, e_u^2, e_v^2, e_b^2) >> \\ &:= \int_X (e_h^1 \wedge f_h^2 + e_h^2 \wedge f_h^1 + e_u^1 \wedge f_u^2 + e_u^2 \wedge f_u^1 + e_v^1 \wedge f_v^2 + e_v^2 \wedge f_v^1) + \int_{\partial X} (e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1) \end{aligned} \tag{5}$$

where $f_h^i, , f_u^i f_v^i \in \Omega^1(X)$, $e_h^i, e_u^i, e_v^i, f_b^i, e_b^i \in \Omega^0(\partial X)$.
Then $\mathcal{D} \subset V \times V^*$ *defined as*

$$\mathcal{D} = \{(f_h, f_u, f_v, f_b, e_h, e_u, e_v, e_b) \in V \times V^* \mid$$
$$\begin{bmatrix} f_h \\ f_u \\ f_v \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} & 0 \\ \mathrm{d} & 0 & -\frac{1}{*h}(\mathrm{d}(*v) + f*) \\ 0 & \frac{1}{*h}(\mathrm{d}(*v) + f*) & 0 \end{bmatrix} \begin{bmatrix} e_h \\ e_u \\ e_v \end{bmatrix}; \begin{bmatrix} f_b \\ e_b \\ e'_v \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{1}{*h} \end{bmatrix} \begin{bmatrix} e_u \mid_{\partial X} \\ e_h \mid_{\partial X} \\ e_v \mid_{\partial X} \end{bmatrix} \tag{6}$$

is a Dirac structure, that is $\mathcal{D} = \mathcal{D}^\perp$, *where* $\perp$ *is with respect to (5).*

Proof. The proof follows the same arguments as in [11], making use of the Stokes' theorem and hence we omit the proof here. Also observe that the skew-symmetric term in (6) does not contribute to the bilinear form (5) □

In terms of shallow water equations (1) the above terms would correspond to

$$\begin{aligned} f_h &= -\frac{\partial}{\partial t} h(x,t), e_h = \delta_h \mathcal{H} = (\frac{1}{2}((*u)(*u) + (*v)(*v)) + g(*h + h_b)) \\ f_u &= -\frac{\partial}{\partial t} u(x,t), e_u = \delta_u \mathcal{H} = (*h)(*u) \\ f_v &= -\frac{\partial}{\partial t} v(x,t), e_v = \delta_v \mathcal{H} = (*h)(*v) \\ f_b &= \delta_u \mathcal{H} \mid_{\partial X}, e_b = -\delta_h \mathcal{H} \mid_{\partial X}, \\ e'_v &= \frac{1}{*h} \delta_v \mathcal{H} \mid_{\partial X} . \end{aligned} \tag{7}$$

The Hamiltonian of the system is given as

$$\mathcal{H} = \int_X \frac{1}{2} \left((*u)h(*u) + (*v)h(*v)\right) + \frac{1}{2} g(*h)h + ghh_b.$$

Substituting (7) into (6), we obtain the shallow water equations (1).

2.3 Energy Balance

Energy balance follows immediately from the power conserving property of the Stokes-Dirac structure, given by

$$\int_X (e_h \wedge f_h + e_u \wedge f_u + e_v \wedge f_v) + \int_{\partial X} e_b \wedge f_b = 0. \tag{8}$$

Hence

$$\begin{aligned} \frac{d}{dt}\mathcal{H} &= \int_X (e_h \wedge f_h + e_u \wedge f_u + e_v \wedge f_v) \\ &= \frac{\partial}{\partial t} h(x,t).\delta_h \mathcal{H} + \frac{\partial}{\partial t} u(x,t).\delta_u \mathcal{H} + \frac{\partial}{\partial t} v(x,t).\delta_v \mathcal{H} \\ &= (\tilde{u}(\frac{1}{2} h \tilde{u}^2 + \frac{1}{2} g\tilde{h}(\tilde{h} + h_b))) \mid_0^L + (\tilde{u}(\frac{1}{2} g\tilde{h}(\tilde{h} + h_b))) \mid_0^L . \end{aligned}$$

The first term in last line of the above expression for energy balance corresponds to the energy flux (the total energy times the velocity) through the boundary and the second term is the work done by the hydrostatic pressure given by pressure times the velocity. Note that the expression for energy balance remains unchanged in presence of coriolis terms.

3 Linear shallow water equations

For the discretization procedure, we consider a simple case of linearizing the system around a given height h_l and zero velocities, i.e. around $(h_l, 0, 0)$. The linear dynamics are derived as follows: We neglect the variations in topography to take a constant mean water depth. Consider small variations of the system around $(h_l, 0, 0)$ and denote

$$\begin{aligned} h(x,t) &= h_l + \epsilon\eta(x,t) \\ u(x,t) &= \epsilon u'(x,t) \\ v(x,t) &= \epsilon v'(x,t). \end{aligned} \tag{9}$$

Substituting (9) in the shallow water equations given by equations (6,7) and taking the limit as $\epsilon \to 0$, we get the following

$$\begin{aligned} \partial_t \eta &= -\mathrm{d}(h_l * u') \\ \partial_t u' &= -\mathrm{d}(g * \eta) + f v' \\ \partial_t v' &= -f u' \end{aligned} \tag{10}$$

For sake of consistent notation throughout, we replace η in (10) by h and u' by u and v' by v. This gives us the following linear shallow water equations

$$\begin{aligned} \partial_t h &= -\mathrm{d}(h_l * u) \\ \partial_t u &= -\mathrm{d}(g * h) + f v \\ \partial_t u &= -f u. \end{aligned}$$

The Hamiltonian in this case is given by

$$\mathcal{H} = \int_X \frac{1}{2}(h_l(u * u + v * v) + \frac{1}{2} g h * h)). \tag{11}$$

The linearized (yet non-constant) Dirac structure of the system would then be given as

$$\begin{bmatrix} f_h \\ f_u \\ f_v \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} & 0 \\ \mathrm{d} & 0 & -\frac{1}{h_l} f* \\ 0 & \frac{1}{h_l} f* & 0 \end{bmatrix} \begin{bmatrix} e_h \\ e_u \\ e_v \end{bmatrix}; \tag{12}$$

together with the boundary conditions.

4 Spatial discretization of the linear shallow water equations

The spatial discretization procedure in the port-Hamiltonian framework essentially consists of two steps. First we discretize the spatially distributed interconnection structure (12). Next we approximate the constitutive relations of the energy storage part.

4.1 Tessellation

At first, we have to tessellate the spatial domain with cells or elements denoted as K_k with spatial manifold $[x_j, x_{j+1}]$ such that

$$\Omega = \{K_k \bigcup_1^N | K_k \cap K_{k'} = \emptyset \text{ if } k \neq k', 1 \leq k, k' \leq N, 0 < x_j < x_{j+1} < L\}.$$

It is convenient to introduce a reference element $\hat{K}$ with the spatial manifold $[-1, 1]$ such that each element K_k is mapped to the reference element $\hat{K}$ using the mapping

$$F_K : \hat{K} \to K_k : x = \frac{1}{2}\big(x_j(1-\zeta) + x_{j+1}(1+\zeta)\big),$$

where ζ is the coordinate of the reference element $\hat{K}$.

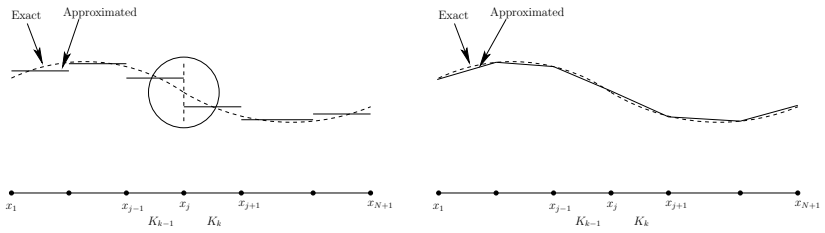

Figure 1. *An illustration of the approximation of a flow variable.*

4.2 Approximation of flows and efforts

Consider a part of the canal between two points x_j and x_{j+1}. We denote the mass flow at point x_j by e_j^B, the Bernoulli function by f_j^B and the boundary term arising due to the velocity component v by e_{vj}^B and similarly at the point x_{j+1}. The relations between the boundary variables $f_j^B, e_j^B, f_{j+1}^B, e_{j+1}^B, f_{vj}^B, e_{vj}^B$ and the efforts e_h, e_u and e_v by

$$\begin{array}{lll} f_j^B = e_u(t,x_j) & e_j^B = e_h(t,x_j) & e_{vj}^B = e_v(t,x_j) \\ f_{j+1}^B = e_u(t,x_{j+1}) & e_{j+1}^B = e_h(t,x_{j+1}) & f_{vj}^B = e_v(t,x_{j+1}). \end{array}$$

The port-Hamiltonian discretization starts by spatially approximating the flow variables (f_h, f_u, f_v) in each element K_k as

$$\begin{aligned} f_h(x,t) &= f_k^h(t)\psi_k^h(x) \\ f_u(x,t) &= f_k^u(t)\psi_k^u(x) \\ f_v(x,t) &= f_k^v(t)\psi_k^v(x), \end{aligned} \tag{13}$$

where the one-forms ψ_k^h, ψ_k^u and ψ_k^v satisfy

$$\int_{K_k} \psi_k^h(x) = 1, \quad \int_{K_k} \psi_k^u(x) = 1, \quad \int_{K_k} \psi_k^u(x) = 1. \tag{14}$$

$\big(f_k^h(t), f_k^u(t), f_k^v(t)\big) = (-dh_k/dt, -du_k/dt, -dv_k/dt)$ are the coefficients of the one forms and $(h_k(t), u_k(t), u_k(t))$ the total depth and velocity in the element K_k at time t. Since the flow variables are approximated per element they can be discontinuous across the nodes of the elements and hence they are multivalued at the nodes as illustrated in Figure 1. In each element K_k, the effort variables $e_h(x,t)$, $e_u(x,t)$ and $e_v(x,t)$ are approximated as

$$\begin{aligned} e_h(x,t) &= e_j^h(t)\phi_j^h(x) + e_{j+1}^h(t)\phi_{j+1}^h(x), \\ e_u(x,t) &= e_j^u(t)\phi_j^u(x) + e_{j+1}^u(t)\phi_{j+1}^u(x) \\ e_v(x,t) &= e_j^v(t)\phi_j^v(x) + e_{j+1}^v(t)\phi_{j+1}^v(x) \end{aligned} \tag{15}$$

where $\phi_j^h(x), \phi_j^u(x), \phi_j^v(x)$ and $\phi_{j+1}^h(x), \phi_{j+1}^u(x), \phi_{j+1}^v(x)$ are the zero-forms, and $(e_j^h(t), e_j^u(t), e_j^v(t))$, $(e_{j+1}^h(t), e_{j+1}^u(t), e_{j+1}^v(t))$ are the values of the efforts (e_h, e_u, e_v) at nodes x_j and x_{j+1}, respectively. The zero-forms always satisfy the property

$$\begin{array}{llll} \phi_j^h(x_j) = 1, & \phi_{j+1}^h(x_j) = 0, & \phi_j^h(x_{j+1}) = 0, & \phi_{j+1}^h(x_{j+1}) = 1 \\ \phi_j^u(x_j) = 1, & \phi_{j+1}^u(x_j) = 0, & \phi_j^u(x_{j+1}) = 0, & \phi_{j+1}^u(x_{j+1}) = 1 \\ \phi_j^v(x_j) = 1, & \phi_{j+1}^v(x_j) = 0, & \phi_j^v(x_{j+1}) = 0, & \phi_{j+1}^v(x_{j+1}) = 1, \end{array} \tag{16}$$

in the element K_k such that the efforts are always continuous across the edges of the elements as illustrated in Figure **??**.

4.3 Compatability of forms

The approximation of flows (13) and efforts (15) with zero and one forms must satisfy the relations described in the Stokes' Dirac structure (12)

$$
\begin{aligned}
f_k^h(t)\psi_k^h(x) &= e_j^u(t)\mathrm{d}\phi_j^u(x) + e_{j+1}^u(t)\mathrm{d}\phi_{j+1}^u(x) \\
f_k^u(t)\psi_k^u(x) &= e_j^h(t)\mathrm{d}\phi_j^h(x) + e_{j+1}^h(t)\mathrm{d}\phi_{j+1}^h(x) - \frac{f*}{h_l}(e_j^v(t)\phi_j^v(x) + e_{j+1}^v(t)\phi_{j+1}^v(x)) \\
f_k^v(t)\psi_k^v(x) &= \frac{f*}{h_l}(e_j^u(t)\phi_j^u(x) + e_{j+1}^u(t)\phi_{j+1}^u(x).
\end{aligned} \tag{17}
$$

To satisfy the first line of (17), the one forms $\psi_k^h(x)$ must be compatible with the zero forms $e_j^u(x)$ and $e_{j+1}^u(x)$ such that for every e_j^u, e_{j+1}^u we can find f_k^h. This implies that if we take $e_{j+1}^u = 0$ then the first line of relation (17) holds if and only if

$$
\mathrm{d}\phi_j^u = f_k^h(t)\psi_k^h(x)/e_j^u(t) \neq 0. \tag{18}
$$

Introducing $c(t) = f_k^{h,u}(t)/e_j^{u,h}(t)$, using the conditions (14) and (16), and integrating (18); we obtain $c(t) = -1$. Therefore, $\mathrm{d}\phi_j^u = -\psi_k^h$ determines the zero form ϕ_j^u given the one form ψ_k^h. Similary, $\mathrm{d}\phi_{j+1}^u = \psi_k^h$ determines the zero form ϕ_{j+1}^u given the one form ψ_k^h. As a consequence, we also have some more additional properties on zero and one forms which can be found in [5] and have been listed below.

$$
\begin{aligned}
\phi_j^h(x) + \phi_{j+1}^h(x) &= 1 \\
\phi_j^u(x) + \phi_{j+1}^u(x) &= 1 \\
\int_{K_k}^h \phi_j^h(x)\psi_k^h(x) + \int_{K_k} \phi_{j+1}^h(x)\psi_k^h(x) &= 1 \\
\int_{K_k} \phi_j^u(x)\psi_k^u(x) + \int_{K_k} \phi_{j+1}^u(x)\psi_k^u(x) &= 1 \\
\int_{K_k}^h \phi_j^h(x)\psi_k^h(x) + \int_{K_k} \phi_{j+1}^u(x)\psi_k^u(x) &= 1
\end{aligned} \tag{19}
$$

The satisfaction of such conditions leads to the following equation relating $f_k^h(t)$ to $e_j^u(t)$ and $e_{j+1}^u(t)$

$$
f_k^h(t) = e_j^u(t) - e_{j+1}^u(t).
$$

The above equation can also be obtained by integrating the first line of (17) over K_k and substituting the conditions on the zero and one forms (14,16). Similar satisfaction of compatibility conditions for second and third lines of (17) gives us the following equations

$$
\begin{aligned}
f_k^u(t) &= e_j^h(t) - e_{j+1}^h(t) - \frac{f}{h_l}(c_1 e_j^v(t) + c_2 e_{j+1}^v(t)) \\
f_k^v(t) &= \frac{f}{h_l}(c_i^{'} e_j^u(t) + c_2^{'} e_{j+1}^u(t).
\end{aligned}
$$

where

$$
c_1 = \int_{K_k} *\phi_j^v(x), c_2 = \int_{K_k} *\phi_{j+1}^v(x), c_1^{'} = \int_{K_k} *\phi_j^u(x), c_2^{'} = \int_{K_k} *\phi_{j+1}^u(x).
$$

For the sake of clarity the argument t is omitted in the rest of the section. The relations describing

the spatially discretized interconnection structure of the part of the canal are given by

$$\begin{bmatrix} e_j^B \\ e_{j+1}^B \\ f_j^B \\ f_{j+1}^B \\ e_{vj}^B \\ e_{vj+1}^B \\ f_k^h \\ f_k^u \\ f_k^v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & - & 0 & 0 & -\frac{f}{h_l}c_1 & -\frac{f}{h_l}c_2 \\ 0 & 0 & \frac{f}{h_l}c_1^{'} & \frac{f}{h_l}c_2^{'} & 0 & 0 \end{bmatrix} \begin{bmatrix} e_j^h \\ e_{j+1}^h \\ e_j^u \\ e_{j+1}^u \\ e_j^v \\ e_{j+1}^v \end{bmatrix}.$$

The net power in the considered part of the canal $[x_j, x_{j+1}]$ (analogous to (8)) is

$$\int_{K_k} [e_h f_h + e_u f_u + e_v f_v] - e_j^B f_j^B + e_{j+1}^B f_{j+1}^B. \tag{20}$$

Using (13,15) and substituting them into the above expression gives

$$P_{K_k}^{net} = [\alpha_k e_j^h + (1-\alpha_k) e_{j+1}^h] f_k^h + [(1-\alpha_k) e_j^u + \alpha_k e_{j+1}^u] f_k^u + [\beta_1 e_j^v + \beta_2 e_{j+1}^v] f_k^v - e_j^B f_j^B + e_{j+1}^B f_{j+1}^B. \tag{21}$$

where $\alpha_k = \int_{K_k} \phi_j^h(x)\psi_k^h(x)$, $\beta_1 = \int_{K_k} \phi_j^v(x)\psi_k^v(x), \beta_2 = \int_{K_k} \phi_{j+1}^v(x)\psi_k^v(x)$. We use the above expression for identifying the port variables in the discretized interconnection structure. The last two terms of (21) imply that the port variables of the incoming port are (f_j^B, e_j) and the port variables of the outgoing port are (f_{j+1}^B, e_{j+1}). The flow variable corresponding to the energy variables are f_k^h, f_k^u, f_k^v and the corresponding effort variables are $\alpha_k e_j^h + (1-\alpha_k) e_{j+1}^h$, $(1-\alpha_k) e_j^u + \alpha_k e_{j+1}^u$ and $\beta_1 e_j^v + \beta_2 e_{j+1}^v$. Thus we define

$$\begin{aligned} e_k^h &:= \alpha_k e_j^h + (1-\alpha_k) e_{j+1}^h \\ e_k^u &:= (1-\alpha_k) e_j^h + \alpha_k e_{j+1}^h \\ e_k^v &:= \beta_1 e_j^v + \beta_2 e_{j+1}^v. \end{aligned} \tag{22}$$

In addition to the properties of the zero and one forms (19), we also have the following properties which are crucial in deriving the expression for power balance in the finite-dimensional case.

Proposition 3. *Under the assumptions that $\psi_k^u = \psi_k^v$, the constants $\alpha_k, \alpha_{k'}, \beta_1, \beta_2, c_1, c_2, c_1', c_2'$ satisfy*

$$\alpha_{k'} c_1 = \beta_1 c_1^{'}, \qquad \alpha_{k'} c_2 = \beta_2 c_1^{'}, \alpha_k c_1 = \beta_1 c_2^{'}, \alpha_k c_1 = \beta_2 c_2^{'}.$$

where $\alpha_{k'} = \int_{K_k} \phi_{j+1}^h(x)\psi_k^h(x)$,

Proof. The proof follows from the compatibility conditions of the zero and one forms in (17) together with the properties listed in (19). □

The net expression for power (20)now becomes

$$P_k^{net} := f_k^h e_k^k + f_k^u e_k^u + f_k^v e_k^v - e_j^B f_j^B + e_{j+1}^B f_{j+1}^B.$$

Now by substituting

$$e_j^h = e_j^B, e_{j+1}^h = e_{j+1}^B, e_j^u = f_j^B, e_{j+1}^u = f_{j+1}^B, e_j^v = f_{vj}^B, e_{j+1}^v = e_{vj}^B,$$

yields

$$\begin{bmatrix} -1 & 0 & 0 & \alpha_k & \alpha_{k'} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & \frac{f}{h_l}c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_k^h \\ e_k^u \\ e_k^v \\ e_j^B \\ e_{j+1}^B \\ e_{j+1}^v \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_k & \alpha_k & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_1 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{f}{h_l}c_1 \\ 0 & 0 & 1 & -\frac{f}{h_l}c_1^{'} & -\frac{f}{h_l}c_2^{'} & 0 \end{bmatrix} \begin{bmatrix} f_k^h \\ f_k^u \\ f_k^v \\ f_j^B \\ f_{j+1}^B \\ f_{j+1}^v \end{bmatrix} = 0. \quad (23)$$

The above equation represents the spatially discretized interconnection structure, abbreviated as

$$D_k = \{(f^k, e^k) \in \mathbb{R}^{12} : E_k e_k + F_k f_k = 0\}.$$

It can easily be shown that the above subspace D_k is a Dirac structure with respect to the bilinear form

$$<< (f_1^k, e_1^k), (f_2^k, e_2^k) >> := < e_1^k, f_2^k > + < e_2^k, f_1^k > . \quad (24)$$

4.4 Approximation of the energy part

For the discretization of the energy (the Hamiltonian (11)) part we proceed as follows: The flow variables f_h, f_u and f_v and the energy variables h, u and v are one-forms. Since f_h, f_u and f_v are approximated by (13) and are related to h, u and v by (15), it is consistent to approximate h,u and v on K_k in the same way by

$$h(x,t) = h_k(t)\psi_k^h(x),\ u(x,t) = u_k(t)\psi_k^u(x),\ v(x,t) = v_k(t)\psi_k^u(x), \quad (25)$$

where

$$-\frac{dh_k(t)}{dt} = f_k^h(t), -\frac{du_k(t)}{dt} = f_k^u(t), -\frac{dv_k(t)}{dt} = f_k^v(t). \quad (26)$$

Here h_k represents the total amount of water in the considered part of the canal and u_k, v_k the average velocities of the same part of the canal. The kinetic energy as a function of the energy variables u and v is given by

$$\int_{K_k} \frac{1}{2}[(*u(x,t))h_l(u(x,t)) + (*v(x,t))h_l(v(t,x))].$$

Approximation of the infinite-dimensional energy variables u and v by (26) means that we restrict the infinite-dimensional space of one-forms $\psi^1(K_k)), \psi^1(K_k)$ to its one-dimensional subspace spanned by ψ_k^u, ψ_k^v. This leads to the approximation of the kinetic energy of the considered part of the canal by

$$H_k^{u,v}(u_k, v_k) = \frac{1}{2}(C_1 h_l u_k^2 + C_2 h_l v_k^2), \text{where } C_1 = \int_{K_k} (*\psi_k^u(x))\psi_k^u(x), C_2 = \int_{K_k} (*\psi_k^v(x))\psi_k^v(x).$$

Note that this is nothing else than the restriction of the kinetic energy function to the one dimensional subspace of $\Omega^1(Z_{ab})$. Similarly the potential energy is approximated by

$$H_k^h(h_k) = \frac{C_3}{2} g h_k^2, \text{where } C_3 = \int_{K_k} (*\psi_k^h(x))\psi_k^h(x).$$

Therefore, the total energy in the considered part of the canal is approximated by

$$H_k(h_k, u_k, v_k) = H_k^{u,v}(u_k, v_k) + H_k^h(h_k) = \frac{1}{2}\left(C_1 h_l u_k^2 + C_2 h_l v_k^2 + C_3 g h_k^2\right).$$

Next, in order to describe the discretized dynamics, we equate the discretized effort variables e_k^h, e_k^u, e_k^v of the discretized interconnection structure defined in (22) with co-energy variables corresponding to the total approximated energy H_k of the considered part of the canal

$$e_k^h = \frac{\partial H(h_k,u_k,v_k)}{\partial h_k}(t) = gC_3h_k,\ e_k^u = \frac{\partial H(h_k,u_k,v_k)}{\partial u_k}(t) = C_1h_lu_k,\ e_k^v = \frac{\partial H(h_k,u_k,v_k)}{\partial v_k}(t) = C_2h_lv_k. \tag{27}$$

The equations (23) (the interconnected structure) together with (26),(27) represent a finite-dimensional model of the shallow water equations with a non-constant Stokes-Dirac structure. To sum up we obtain the following set of DAEs for a single lump of the finite-dimensional model

$$\begin{aligned}
&-\tfrac{dh_k}{dt} = h_lu\mid_j - h_lu\mid_{j+1}\\
&-\tfrac{du_k}{dt} = gh\mid_j - gh\mid_{j+1} - \left(\tfrac{f}{h_l}(c_1h_lv\mid_j + c_2h_lv\mid_{j+1})\right)\\
&-\tfrac{dv_k}{dt} = \tfrac{f}{h_l}(c_1'h_lu\mid_j + c_2'h_lu\mid_{j+1})\\
&C_3gh_k = \alpha_k(gh\mid_j) + \alpha_{k'}(gh\mid_{j+1})\\
&C_1h_lu_k = \alpha_{k'}(h_lu\mid_j) + \alpha_k(h_lu\mid_{j+1})\\
&C_2h_lv_k = \beta_1(h_lv\mid_j) + \beta_2(h_lv\mid_{j+1}).
\end{aligned} \tag{28}$$

Spatial discretization of the entire system

So far we have seen the spatial discretization of a single lump in the canal. We now focus on the discretization of the entire system. The canal is split into N parts (which are interconnected to each other) with $N+1$ nodes. The $k-$th part is discretized as explained in the previous subsections. The resulting model consists of N submodels each of them representing a port-Hamiltonian system. Since a power conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system, the total discretized system is also a port-Hamiltonian system, whose interconnection structure is given by the composition of the N Dirac structures on (K_k), while the total Hamiltonian is the sum of individual Hamiltonians as

$$H(h,u,v) = \sum_{k=1}^{N}[C_{1k}h_lu_k + C_{2k}h_lv_k + C_3gh_k^2].$$

Here $h = (h_1, h_2, ..., h_N)^T$ are the discretized heights and $u = (u_1, u_2, ..., u_N)$ and $v = (v_1, v_2, ..., v_N)$ are the discretized velocities. The total discretized model still has two ports. The port (f_1^B, e_1^B) is the incoming port and the port (f_{N+1}^B, e_{N+1}^B) is the outgoing port, resulting in the energy balance of the discretized model

$$\frac{dH(h(t),u(t),v(t))}{dt} - e_1^Bf_1^B + e_{N+1}^Bf_{N+1}^B = 0.$$

In case of a periodic domain

$$\frac{dH(h(t),u(t),v(t))}{dt} = 0,$$

hence the energy is conserved in the discretized model.

5 The linear state space model

In this section we write the discretized system in the input state output model and discuss its interconnection properties. To simplify the model we use the following choices for the approximating zero and one forms. The zero-forms are approximated as constant density functions, i.e.

$$\psi_k^{h,u,v} = \frac{1}{\Delta x},$$

and the zero-forms as linear splines, i.e.

$$\phi_j^{h,u,v} = \frac{x_{j+1}-x}{x_{j+1}-x_j},\quad \phi_{j+1}^{h,u,v} = \frac{x-x_j}{x_{j+1}-x_j}.$$

This choice would result in the following input output model of a single lump of the system.

$$\begin{bmatrix} f_k^h \\ f_k^u \\ f_k^v \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -\frac{f}{h_l} \\ 0 & \frac{f}{h_l} & 0 \end{bmatrix} \begin{bmatrix} e_k^h \\ e_k^u \\ e_k^v \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -f_{j+1}^B \\ e_j^B \end{bmatrix}; \begin{bmatrix} e_{j+1}^B \\ f_j^B \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} e_k^h \\ e_k^u \\ e_k^v \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -f_{j+1}^B \\ -e_j^B \end{bmatrix}, \tag{29}$$

5.1 Discussion

It is interesting to note here that if we consider a periodic domain then, for the overall system to be well-posed, we should only consider odd number of spatial lumps. The reason for this can be explained as follows: Consider a periodic domain with only two lumps, with the input state output model (29) of each of the lump in the following (linear port-Hamiltonian) form

$$\begin{aligned} \dot{x}_i &= A_i Q_i x + B_i u_i \\ y_i &= B_i^T Q_i x + D_i u_i, \ i = 1, 2. \end{aligned}$$

In terms of the discretized shallow water equations

$$A_i = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -\frac{f}{h_l} \\ 0 & \frac{f}{h_l} & 0 \end{bmatrix}, Q_i = \begin{bmatrix} g & 0 & 0 \\ 0 & h_l & 0 \\ 0 & 0 & h_l \end{bmatrix}, B_i = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, D_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We know from system theory that an interconnection of the two such lumps, in the standard plant controller interconnection constraints

$$u_1 = -y_2, \ u_2 = y_1, \text{where } u_1 \begin{bmatrix} -f_2^B \\ e_1^B \end{bmatrix}, u_2 = \begin{bmatrix} -f_3^B \\ e_2^B \end{bmatrix}, y_1 \begin{bmatrix} e_2^B \\ f_1^B \end{bmatrix}, y_2 = \begin{bmatrix} e_3^B \\ f_2^B \end{bmatrix} \tag{30}$$

where is well-posed if and only if the following two conditions are satisfied

$$\det[I + D_1 D_2] \text{and } \det[I + D_2 D_1] \neq 0. \tag{31}$$

It can easily be seen that for the input state output model of the shallow water equations, the above conditions are not satisfied and hence we cannot interconnect the two spatially distributed lumps interconnected by the constraints (30). Same is the case if we consider any even number of lumps. This obstacle is eliminated if we consider odd number of lumps. Hence its crucial that in our analysis, for the system to be well posed, we should consider only an odd number of lumps for a periodic domain.

6 Preliminary numerical results

The spatial discretization of one dimensional linear and nonlinear shallow water equations using port-Hamiltonian frame work are presented. The discretization typically consists of a set of ordinary differential and algebraic equations in which we seek for the solution of flows and efforts ("energy fluxes") numerically. We have attempted to solve the resulting ordinary differential equations using explicit time-stepping schemes like Euler-forward and Runge-Kutta methods. We found that these time-stepping schemes are numerically unstable and hence we use the Crank-Nicholson time-stepping scheme. Investigation on explicit time-stepping schemes and their stability analysis is beyond the scope of this study. Below we give an analytical proof of energy conservation in time for the discretized linear shallow water equations.

Proposition 4. *Consider the discretization of the linear shallow water equations (12), which for the $k-th$ lump would be as follows*

$$\begin{array}{l} -\frac{dh_k}{dt} = h_l u\mid_j - h_l u\mid_{j+1} \\ -\frac{du_k}{dt} = gh\mid_j - gh\mid_{j+1} - \left(\frac{f}{h_l}(c_1 h_l v\mid_j + c_2 h_l v\mid_{j+1})\right) \\ -\frac{dv_k}{dt} = \frac{f}{h_l}(c_1' h_l u\mid_j + c_2' h_l u\mid_{j+1}) \\ \frac{gh_k}{\Delta x} = \alpha_k(gh\mid_j) + \alpha_{k'}(gh\mid_{j+1}) \\ \frac{h_l u_k}{\Delta x} = \alpha_{k'}(h_l u\mid_j) + \alpha_k(h_l u\mid_{j+1}) \\ \frac{h_l v_k}{\Delta x} = \beta_1(h_l u\mid_j) + \beta_2(h_l u\mid_{j+1}). \end{array} \tag{32}$$

In case of a uniform mesh Δx would be constant. The total energy is given by

$$H(h_k, u_k, v_k) = \frac{1}{2\Delta x}\sum_{k=1}^{N}[h_l(u_k^2 + v_k^2) + gh_k^2]. \tag{33}$$

For a periodic boundary the efforts can be calculated by the following expression

$$e_j = ... + f_{k-3} - f_{k-2} + f_{k-1_i} + f_k - f_{k+1} + ...; k = 1, ..., N. \tag{34}$$

This expression is obtained by solving the last two equations of (32) simultaneously. Therefore

$$e_j - e_{j+1} = 2(... - f_{k-2} + f_{k-1} - f_{k+1}...) \text{ and } e_j - e_{j+1} = 2f_k \tag{35}$$

where

$$e_j = \begin{bmatrix} gh\mid_j \\ h_l u\mid_j \\ h_l v\mid_j \end{bmatrix}; \quad f_k = \begin{bmatrix} \frac{gh_k}{\Delta x} \\ \frac{h_l u_k}{\Delta x} \\ \frac{h_l v_k}{\Delta x} \end{bmatrix}. \tag{36}$$

Time stepping scheme: As stated before, we use the Crank-Nicholson time stepping scheme. The time discretized equations take the form

$$\begin{array}{l} h_k^{t_{n+1}} - h_k^{t_n} = -\frac{\Delta t}{2}[(h_l u\mid_j^{t_{n+1}} - h_l u\mid_{j+1}^{t_{n+1}}) - (h_l u\mid_j^{t_n} - h_l u\mid_{j+1}^{t_n})] \\ u_k^{t_{n+1}} - u_k^{t_n} = -\frac{\Delta t}{2}[(gh\mid_j^{t_{n+1}} - gh\mid_{j+1}^{t_{n+1}}) - \left(\frac{f}{h_l}(h_l v\mid_j^{t_{n+1}} + h_l v\mid_{j+1}^{t_{n+1}})\right) \\ \qquad\qquad - (gh\mid_j^{t_n} - gh\mid_{j+1}^{t_n}) + \left(\frac{f}{h}(h_l v\mid_j^{t_n} + h_l v\mid_{j+1}^{t_n})\right)] \\ v_k^{t_{n+1}} - v_k^{t_n} = -\frac{\Delta t}{2}[\left(\frac{f}{h_l}(h_l u\mid_j^{t_{n+1}} + h_l u\mid_{j+1}^{t_{n+1}})\right) - \left(\frac{f}{h_l}(h_l u\mid_j^{t_n} + h_l u\mid_{j+1}^{t_n})\right)]. \end{array} \tag{37}$$

In this case the energy of the system is conserved in time.

Proof. Substituting the expressions for efforts from Equation (4) we get

$$\begin{aligned} h_k^{t_{n+1}} - h_k^{t_n}(t_n) &= -h_l\Delta t[(... - u_{k-2}^{t_{n+1}} + u_{k-1}^{t_{n+1}} - u_{k+1}^{t_{n+1}} + ...) - (... - u_{k-2}^{t_n} + u_{k-1}^{t_n} - u_{k+1}^{t_n} + ...), \\ u_k^{t_{n+1}} - u_k^{t_n} &= -g\Delta t[(... - h_{k-2}^{t_{n+1}} + h_{k-1}^{t_{n+1}} - h_{k+1}^{t_{n+1}} + ...) - (... - h_{k-2}^{t_n} + h_{k-1}^{t_n} - h_{k+1}^{t_n} + ...)] \\ &\quad + \Delta t[\frac{fh_l}{h_l^{t_{n+1}}}(v_k^{t_{n+1}} - v_k^{t_n})] \end{aligned} \tag{38}$$

$$v_k^{t_{n+1}} - v_k^{t_n} = -\Delta t[\frac{fh_l}{h_l^{t_{n+1}}}(u_k^{t_{n+1}} - u_k^{t_n})]. \tag{39}$$

Multiplying the first of the above equations by $g\frac{h_k(t_{n+1})+h_k(t_n)}{2\Delta x}$, we can write the discretized potential energy as

$$\begin{aligned} \sum_{k=1}^{N}\frac{g\left[(h_k^{t_{n+1}})^2 - (h_k^{t_n})^2\right]}{2\Delta x} &= -\sum_{k=1}^{N}\left\{\frac{gh_l\Delta t}{(\Delta x)^2}\{... - (u_{k-2}^{t_{n+1}} + u_{k-2}^{t_n})(h_k^{t_{n+1}} + h_k^{t_n}) \right. \\ \left. + (u_{k-1}^{t_{n+1}} + u_{k-1}^{t_n})(h_k^{t_{n+1}} + h_k^{t_n}) - (u_{k+1}^{t_{n+1}} + u_{k+1}^{t_n})(h_k^{t_{n+1}} + h_k^{t_n}) + ...\right\}. \end{aligned}$$

Figure 2. *a) Space-time profile of the free surface perturbation obtained from the numerical scheme of the port-Hamiltonian discretization of the linear shallow water equations with amplitude* $A = 0.01$*,* $\Delta x = 1/101$ *and* $\Delta t = T/10$*. b) Plot of the energy* $H(t) = \int_X \frac{1}{2}(h_l u^2 + gh^2 dx$ *versus time.*

Similarly, multiplying the second and third of the equation by $h_l \frac{u_k(t_{n+1})+u_k(t_n)}{2\Delta x}$ and $h_l \frac{v_k(t_{n+1})+v_k(t_n)}{2\Delta x}$ respectively and adding them, the expression for the discretized kinetic energy takes the following form

$$\sum_{k=1}^{N}\left\{\frac{h_l\left[(u_k^{t_{n+1}})^2-(u_k^{t_n})^2\right]}{(\Delta x)^2}+\frac{h_l\left[(v_k^{t_{n+1}})^2-(v_k^{t_n})^2\right]}{(\Delta x)^2}\right\}=-\sum_{k=1}^{N}\left\{\frac{gh_l\Delta t}{(\Delta x)^2}\left\{...-(h_{k-2}^{t_{n+1}}+h_{k-2}^{t_n})(u_k^{t_{n+1}}+u_k^{t_n})\right.\right.$$
$$\left.\left.+(h_{k-1}^{t_{n+1}}+h_{k-1}^{t_n})(u_k^{t_{n+1}}+u_k^{t_n})-(h_{k+1}^{t_{n+1}}+h_{k+1}^{t_n})(u_k^{t_{n+1}}+u_k^{t_n})+...\right\}\right. . \tag{40}$$

Since we have a periodic domain and also since the summation is over the entire domain, we can rewrite the right hand side of (40), by interchanging the indices, as follows

$$\sum_{k=1}^{N}\frac{g\left[(h_k^{t_{n+1}})^2-(h_k^{t_n})^2\right]}{2\Delta x}=-\sum_{k=1}^{N}\left\{\frac{gh_l\Delta t}{(\Delta x)^2}\left\{...-(u_k^{t_{n+1}}+u_k^{t_n})(h_{k+2}^{t_{n+1}}+h_{k+2}^{t_n})\right.\right.$$
$$\left.\left.+(u_k^{t_{n+1}}+u_k^{t_n})(h_{k+1}^{t_{n+1}}+h_{k+1}^{t_n})-(u_k^{t_{n+1}}+u_k^{t_n})(h_{k-1}^{t_{n+1}}+h_{k-1}^{t_n})+...\right\}\right. . \tag{41}$$

Now adding (40) and (41) we have that, the rate of change of total energy

$$\sum_{k=1}^{N}\frac{h_l\left[(u_k^{t_{n+1}})^2-(u_k^{t_n})^2\right]}{(\Delta x)^2}+\frac{h_l\left[(v_k^{t_{n+1}})^2-(v_k^{t_n})^2\right]}{(\Delta x)^2}+\frac{g\left[(h_k^{t_{n+1}})^2-(h_k^{t_n})^2\right]}{2\Delta x}=0, \tag{42}$$

which implies that

$$\sum_{k=1}^{N}H_k^{t_{n+1}}-H_k^{t_n}=0,$$

and hence we prove that the total energy of the discretized system is conserved in time. □

6.1 Harmonic wave type solution

Our preliminary numerical results are for the very simple case by setting v, f in (12) to zero.We consider the following harmonic wave type solution of one dimensional linear shallow water equation in a domain $[0, L]$:

$$h(x,t)=h_l+\eta(x,t),\quad \eta(x,t)=A\sin(kz+\omega t),\quad \text{and } u(x,t)\quad =\left(\frac{-Agk}{\omega}\right)\sin(kx+\omega t). \tag{43}$$

where A is the amplitude, $k = 2\pi m/L$ the wave number, ω the actual frequency, h_l the mean free surface depth, $a^2 = gh_l$, $\omega^2 = a^2k^2$ the dispersion relation and m an integer. We have initialized the exact solution (43) in the linear numerical code with parameters $L = 1$, $m = 1$, $g = 1$, and $h_l = 1$; and simulated the waves for one time period $T = 2\pi/\omega$. Figure 2(a) shows the space-time profile of the free surface perturbation around the mean water depth from $t = 0$ to 1.0. The numerical discretization not only conserves mass but also energy as shown in Figure 2(b).

7 Conclusions

In this paper we have shown a discretization prodecure for shallow water equations, modelled as port-Hamiltonian systems which are now defined with respect to a *non-constant* Stokes-Dirac structure. It is observed that a key feature of this methodology is that the discretized system is again a port-Hamiltonian system. The advantages of it are that the physical properties of the distributed parameter model can be translated to the lumped parameter approximation, while retaining the boundary conditions. This means that the approximated model can be interconnected to other systems in the same way as was for the distributed parameter model. We also show that in the linear case, the energy is conserved in time. Towards the end, we present some numerical simulations to show the properties of the approximated model.

Current research focuses on extending these results to the non linear case and also consider cases with higher dimensional spatial domains.

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